Bernoulli's Law

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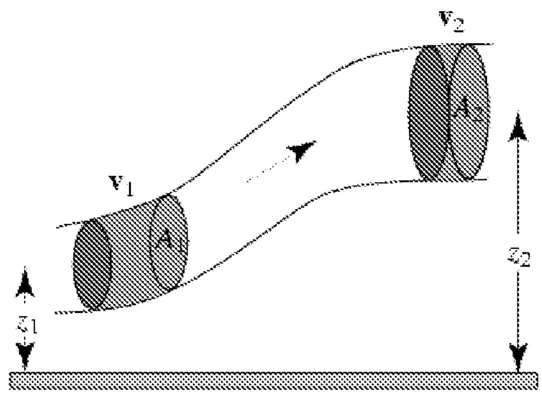
Bernoulli's law describes the behavior of a fluid under varying conditions of flow and height. It states

$$P + \tfrac{1}{2}\rho v^2 + \rho g h = [\text{constant}], \quad (1)$$

where P is the static pressure (in Newtons per square meter), ρ is the fluid density (in kg per cubic meter), v is the velocity of fluid flow (in meters per second) and h is the height above a reference surface. The second term in this equation is known as the dynamic pressure. The effect described by this law is called the Bernoulli effect, and (1) is sometimes known as Bernoulli's equation.



For a heuristic derivation of the law, picture a pipe through which and ideal fluid is flowing at a steady rate. Let W denote the work done by applying a pressure P over an area A, producing an offset of Δl, or volume change of ΔV. Let a subscript 1 denote fluid parcels at an initial point down the pipe, and a subscript 2 denote fluid parcels further down the pipe. Then the work done by pressure force

$$dW = P\,dV \quad (2)$$

at points 1 and 2 is

$$\Delta W_1 = P_1 A_1 \Delta l_1 = P_1 \Delta V \quad (3)$$

$$\Delta W_2 = P_2 A_2 \Delta l_2 = P_2 \Delta V \quad (4)$$

and the difference is

$$\Delta W \equiv \Delta W_1 - \Delta W_2 = P_1 \Delta V - P_2 \Delta V. \quad (5)$$

Equating this with the change in total energy (written as the sum of kinetic and potential energies gives

$$\Delta W = \Delta K + \Delta U \quad (6)$$

$$= \tfrac{1}{2}\Delta m v_2^2 - \tfrac{1}{2}\Delta m v_1^2 + \Delta m g z_2 - \Delta m g z_1.$$

Equating (6) and (5),

$$\tfrac{1}{2}\Delta m v_2^2 - \tfrac{1}{2}\Delta m v_1^2 + \Delta m g z_2 - \Delta m g z_1 = P_1 \Delta V - P_2 \Delta V, \qquad (7)$$

which, upon rearranging, gives

$$\frac{\Delta m v_1^2}{2\Delta V} + \frac{\Delta m g z_1}{\Delta V} + P_1 = \frac{\Delta m v_2^2}{2\Delta V} + \frac{\Delta m g z_2}{\Delta V} + P_2, \qquad (8)$$

so writing the density as $\rho = m/V$ then gives

$$\tfrac{1}{2}\rho v^2 + \rho g z + P = [\text{const}]. \qquad (9)$$

This quantity is constant for all points along the streamline, and this is Bernoulli's theorem, first formulated by Daniel Bernoulli in 1738. Although it is not a new principle, it is an expression of the law of conservation of mechanical energy in a form more convenient for fluid mechanics.

A more rigorous derivation proceeds using the one-dimensional Euler's equation of inviscid motion,

$$\rho u \frac{\partial u}{\partial l} = -\frac{\partial P}{\partial l} \qquad (10)$$

along a streamline, where *u* is used for speed instead of *v* (a common convention in fluid mechanics). Integrating gives

$$\rho u \, du = -dP \qquad (11)$$

$$\tfrac{1}{2}\rho u^2 + P = [\text{const}]. \qquad (12)$$

In a gravitational field, this becomes

$$\tfrac{1}{2}\rho u^2 + \rho g z + P = [\text{const}]. \qquad (13)$$

However, if the flow has zero vorticity, then

$$\mathbf{u} \cdot \nabla \mathbf{u} = \tfrac{1}{2}\nabla(\mathbf{u} \cdot \mathbf{u}) - \mathbf{u} \times (\nabla \times \mathbf{u}) = \tfrac{1}{2}\nabla(u^2), \qquad (14)$$

but

$$\mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho}\nabla P, \qquad (15)$$

so, for incompressible flow,

$$\nabla(\tfrac{1}{2}\rho u^2 + P) = 0 \qquad (16)$$

$$\frac{1}{2}\rho u^2 + P = [\text{const}] \quad (17)$$

throughout the entire fluid.

SEE ALSO: Bernoulli Effect, d'Alembert's Paradox, Dynamic Pressure, Kutta-Zhukovski Theorem, Lift, Lift Coefficient, Lift Force, Static Pressure

Dynamic Pressure

Dynamic pressure is the component of fluid pressure that represents fluid kinetic energy (i.e., motion), while static pressure represents hydrostatic effects, so

$$P_{\mathrm{total}} = P_{\mathrm{dynamic}} + P_{\mathrm{static}}.$$

The dynamic pressure of a fluid with density ρ and speed u is given by

$$P_{\mathrm{dynamic}} \equiv \tfrac{1}{2}\rho u^2,$$

which is precisely the second term in Bernoulli's law.

SEE ALSO: Bernoulli's Law, Pressure, Static Pressure

Bernoulli Equation

A non-turbulent, perfect, compressible, and barotropic fluid undergoing steady motion is governed by the Bernoulli Equation:

$$\frac{V^2}{2g} + z + \frac{\tilde{P}}{g} = C\,(streamline)$$

where g is the gravity acceleration constant (9.81 m/s^2; 32.2 ft/s^2), V is the velocity of the fluid, and z is the height above an arbitrary datum. C remains constant along any streamline in the flow, but varies from streamline to streamline. If the flow is irrotational, then C has the same value for all streamlines.

The function $\tilde{p}$ is the "pressure per density" in the fluid, and follows from the barotropic equation of state, $p = p(\mathrm{r})$.

For an incompressible fluid, the function $\tilde{p}$ simplifies to p/r, and the incompressible Bernoulli Equation becomes:

$$\frac{V^2}{2g} + z + \frac{p}{\rho g} = C$$

Derivation from Navier-Stokes

The Navier-Stokes equation for a perfect fluid reduce to the **Euler Equation**:

$$-\vec{\nabla} p + \rho \mathbf{b} = \rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\vec{\nabla}\mathbf{v}\right)$$

Rearranging, and assuming that the body force **b** is due to gravity only, we can eventually integrate over space to remove any vector derivatives,

$$-\frac{1}{\rho}\tilde{\nabla}p - g\mathbf{i}_z = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\tilde{\nabla}\mathbf{v}$$

$$-\frac{d\tilde{P}}{dp}\tilde{\nabla}p - \tilde{\nabla}(gz) = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\tilde{\nabla}\mathbf{v}$$

$$0 = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\tilde{\nabla}\mathbf{v} + \tilde{\nabla}\left(\tilde{P} + gz\right)$$

$$C(t, streamline) = \int \frac{\partial v_i}{\partial t}\, dx_i + \frac{1}{2}V^2 + \tilde{P} + gz$$

If the fluid motion is also steady (implying that all derivatives with respect to time are zero), then we arrive at the Bernoulli equation after dividing out by the gravity constant (and absorbing it into the constant *C*),

$$\frac{V^2}{2g} + z + \frac{\tilde{P}}{g} = C\left(streamline\right)$$

Note that the fluid's barotropic nature allowed the following chain rule application,

$$\tilde{\nabla}\tilde{P} = \frac{d\tilde{P}}{dp}\tilde{\nabla}p = \frac{1}{\rho}\tilde{\nabla}p$$

with the "**pressure per density**" function $\tilde{P}$ defined as,

$$\tilde{P}(p) = \int_{p_0}^{p} \frac{d\tilde{p}}{\rho}$$